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On a Functional Equation Arising in the Theory of Queues

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On a Functional Equation Arising in the Theory of Queues

Herman Hanisch and Warren M. Hirsch

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We consider in this paper the following functional equation which occurs in the theory of queues :

$$(1)\quad \phi(x) = \int_{[0,\infty)} \int_{[0,x]} \phi(x-t+\tau)\,dF(t)\,dG(\tau) + (1-\alpha)(1-F(x)) \int_{[0,\infty)} \phi(t)\,dG(t) .$$

Here α is any real number in the interval $0 < \alpha \leq 1$, and $F(x)$ and $G(x)$ are arbitrary right-continuous distribution functions bounded by 0 and 1, and such that $F(x) = G(x) = 0$ for $x < 0$. Equation (1), with $\alpha = 1$, is satisfied, for example, by the probability $\phi(x)$ that infinitely many customers will be served during the busy period of a single server queue in which (1) the first customer has service time x, (2) the service times of the remaining customers are independent, identically distributed random variables having distribution function G, and (3) the inter-arrival times are independent, identically distributed random variables with distribution function F. It is also satisfied, with α arbitrary, by the probability $\phi(x)$ of serving infinitely many members of an infinite moving queue whose first member is initially a distance x from a barrier at which the server suffers a risk α of absorption whenever a queue element arrives unserviced [1,2].

If we make the transformation

$$(2)\quad \mu(x) = \phi(x) - (1-\alpha) \int_{[0,\infty)} \phi(t)\,dG(t) ,$$

then μ satisfies the equation

$$(3)\quad \mu(x) = \int_{[0,\infty)} \int_{[0,x]} \mu(x-t+\tau)\,dF(t)\,dG(\tau) .$$

Integrating both sides of (2) with respect to the measure associated with G, solving for $\int_{[0,\infty)} \phi(t)dG(t)$, and substituting the result into (2) we find that

$$(4) \quad \phi(x) = \mu(x) + \frac{1-\alpha}{\alpha} \int_{[0,\infty)} \mu(t)dG(t) .$$

Equations (2) and (4) establish a biunique correspondence between the solutions of (1) and (3) such that corresponding functions differ only by a constant. Accordingly, the correspondence preserves properties such as monotonicity, boundedness, and continuity. Moreover, as is seen from (4), the solution $\mu(x) \equiv 0$ of (3) corresponds to the solution $\phi(x) \equiv 0$ of (1). The relation between ϕ and μ makes it possible in the sequel to reduce the study of equation (1) to that of equation (3).

In the probabilistic context in which (1) (or equivalently (3)) arises, only solutions that are non-decreasing and take values in the unit interval are of interest. Confining attention to the class $\mathcal{C}$ of all such solutions, by a combination of probabilistic and analytical arguments it is proved in [2] that if F and G do not both degenerate at the same point, $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t) = 1$, and $\int_{[0,\infty)} tdG(t) < \infty$, then a necessary and sufficient condition that the only function in $\mathcal{C}$ is the trivial one, $\mu(x) \equiv 0$, is that

$$(5) \quad \int_{[0,\infty)} tdG(t) \leq \int_{[0,\infty)} tdF(t) .^1$$

---

[1] If there is a point t_0 such that $F(t_0)-F(t_0-0) = G(t_0)-G(t_0-0)=1$, then any function μ satisfies (3).

In the argument leading to this result the monotonicity, non-negativity, and boundedness of μ play decisive roles, and it is natural to wonder about the consequences of abandoning these assumptions.

As the following example shows the restriction to bounded solutions is essential not only to the method of proof but to the result itself. Let $c > 1$ be arbitrary,

$$F(t) = \begin{cases} 0 & , \quad t < c \\ 1 & , \quad t \geq c \ , \end{cases}$$

and

$$G(t) = \begin{cases} 0 & , \quad t < c-1 \\ \frac{1}{2} & , \quad c-1 \leq t < c+1 \\ 1 & , \quad t \geq c+1 \end{cases}$$

Evidently (5) is satisfied. Yet, as is straightforward to verify, (3) has the non-trivial (unbounded) solution

$$\mu(x) = \begin{cases} 0 & , \quad 0 \leq x < c \\ j & , \quad c+j-1 \leq x < c+j \quad , \quad j = 1,2, \ldots \end{cases}$$

We are thus led to consider the question of the possible existence of non-trivial, non-monotonic bounded solutions. Let $\mathcal{C}^*$ denote the class of all bounded solutions of (3). Does $\mathcal{C}^*$ contain only the trivial solution if F and G satisfy (5) ? Although the theorem below does not provide a definitive answer, it "almost" does so. For brevity, we put $\mu_F = \int_{[0,\infty)} t dF(t)$ and $\mu_G = \int_{[0,\infty)} t dG(t)$. We use the notation "a.e. $[\lambda]$" to mean

"everywhere except possibly on a set of Lebesgue measure 0".

THEOREM 1. If F and G do not degenerate at the same point (non-degeneracy assumption), $\lim_{t \to \infty} F(t) = \lim_{t \to \infty} G(t) = 1$, and $\mu_G < \infty$, then (5) is a necessary and sufficient condition that $\mathcal{C}^*$ is a subset of functions of the form $\mu(x) = 0$ a.e.$[\lambda]$.

REMARK. This is equivalent to asserting that, under the hypotheses of the theorem, the only bounded solutions of (1) are functions ϕ of the form $\phi(x) =$ const. a.e. $[\lambda]$.

Proof : Assume that (5) holds. Define

$$\Gamma(x) = \int_{[0,\infty)} |\mu(x+t)|\,dG(t).$$

Interchanging the order of integration in (3) it follows that

$$|\mu(x)| \leq \int_{[0,x]} \Gamma(x-t)\,dF(t) \tag{6}$$

The convolution on the right-hand side of (6) suggests introducing the transforms

$$\tilde{M}(s) = \int_{[0,\infty)} e^{-st} |\mu(t)|\,dt \ ,$$

$$\tilde{\Gamma}(s) = \int_{[0,\infty)} e^{-st}\, \Gamma(t)\,dt \ ,$$

and

$$\tilde{F}(s) = \int_{[0,\infty)} e^{-st}\, dF(t) \ .$$

Then by the convolution theorem for Laplace-Stieltjes transforms we have

$$(7) \qquad \tilde{M}(s) \leq \tilde{\Gamma}(s)\tilde{F}(s) .$$

Letting

$$\tilde{G}(s) = \int_{[0,\infty)} e^{-st}\, dG(t) ,$$

it is convenient to rewrite (8) in the form

$$(9) \qquad \tilde{M}(s) - \tilde{\Gamma}(s)\,\tilde{G}(s) \leq s\,\tilde{\Gamma}(s)\,\frac{\tilde{F}(s)-\tilde{G}(s)}{s} .$$

Since

$$\lim_{s \downarrow 0} \frac{\tilde{F}(s)-\tilde{G}(s)}{s} = \mu_G - \mu_F \leq 0 ,$$

and

$$\lim_{s \downarrow 0} \sup\ s\tilde{\Gamma}(s) \geq 0 ,$$

we have

$$(10) \qquad \lim_{s \downarrow 0} \sup\ (\tilde{M}(s) - \tilde{\Gamma}(s)\,\tilde{G}(s)) \leq 0 .$$

We shall show that the limit as $s \downarrow 0$ of the left-hand side of (10) exists. To this end, we express $\tilde{M}(s)$ in the form

$$\tilde{M}(s) = \int_{[0,\tau]} e^{-st}\, |\mu(t)|dt + \int_{(\tau,\infty)} e^{-st}\, |\mu(t)|dt =$$

$$\int_{[0,\tau]} e^{-st}\, |\mu(t)|dt + e^{-s\tau} \int_{(0,\infty)} e^{-st}\, |\mu(t+\tau)|dt =$$

$$\int_{[0,\infty)} \int_{[0,\tau]} e^{-st}\, |\mu(t)|dt\, dG(\tau) + \int_{[0,\infty)} e^{-s\tau} \int_{(0,\infty)} e^{-st} |\mu(t+\tau)|dt dG(\tau)$$

$$= \int_{[0,\infty)} e^{-st}|\mu(t)|(1-G(t))dt + \int_{[0,\infty)} e^{-s\tau}\int_{(0,\infty)} e^{-st}|\mu(t+\tau)|dtdG(\tau).$$

Since, by Fubini's theorem,

$$\tilde{\Gamma}(s)\tilde{G}(s) = \tilde{G}(s)\int_{[0,\infty)}\int_{(0,\infty)} e^{-st}|\mu(t+\tau)|dt\, dG(\tau),$$

we have

$$(11)\quad \tilde{M}(s) - \tilde{\Gamma}(s)\tilde{G}(s) = \int_{[0,\infty)} e^{-st}|\mu(t)|(1-G(t))dt + \int_{[0,\infty)} (e^{-s\tau} - \tilde{G}(s))\int_{(0,\infty)} e^{-st}|\mu(t+\tau)|dtdG(\tau).$$

Put

$$J_1(s) = \int_{[0,\infty)} e^{-st}|\mu(t)|(1-G(t))dt,$$

$$J_2^{(N)}(s) = \int_{[N,\infty)} (e^{-s\tau} - \tilde{G}(s))\int_{(0,\infty)} e^{-st}|\mu(t+\tau)|dtdG(\tau),$$

and

$$J_3^{(N)}(s) = \int_{[N,\infty)} (e^{-s\tau} - \tilde{G}(s))\int_{(0,\infty)} e^{-st}|\mu(t+\tau)|dt\, dG(\tau).$$

Then, for arbitrary N,

$$(12)\quad \tilde{M}(s) - \tilde{\Gamma}(s)\tilde{G}(s) = J_1(s) + J_2^{(N)}(s) + J_3^{(N)}(s).$$

To estimate $J_2^{(N)}(s)$ we put

$$Q(t) = 1 - G(t),$$

$$\tilde{Q}(s) = \int_{[0,\infty)} e^{-st}Q(t)\, dt, \quad s > 0,$$

and

$$h(s,\tau) = \frac{1-e^{-s\tau}}{s} - \tilde{Q}(s) , \quad s > 0, \quad \tau > 0 .$$

It is easy to verify that

$$\tilde{Q}(s) = \frac{1-\tilde{G}(s)}{s} ,$$

so that

$$(13) \qquad s\, h(s,\tau) = \tilde{G}(s) - e^{-s\tau} .$$

Let $\tau \geq N$ and $0 < s \leq \frac{1}{N}$. Then, using the inequality $1-e^{-x} > x/2$, $0 < x \leq 1$, we have

$$(14)\ h(s,\tau) \geq h(s,N) = \frac{1-e^{-sN}}{s} - \tilde{Q}(s) > \frac{N}{2} - \tilde{Q}(0) =$$

$$\frac{N}{2} - \int_{[0,\infty)} Q(t)dt = \frac{N}{2} - m_G .$$

Moreover, since $1-e^{-x} < x$, $x \neq 0$,

$$(15) \qquad h(s,\tau) \leq \frac{1-e^{-s\tau}}{s} < \tau .$$

It follows from (14) and (15) that, if $N \geq 2m_G$, $\tau \geq N$, and $0 < s \leq 1/N$, then

$$(16) \qquad 0 < sh(s,\tau) < s\tau .$$

Since there is a constant A such that $|\mu(x)| \leq A$, we obtain from (13) and (16) the estimate

$$|J_2^{(N)}(s)| \leq A \int_{[N,\infty)} t dG(t) \text{ for } 0 < s \leq \frac{1}{N} \text{ and } N \geq 2m_G .$$

To obtain an estimate of $J_3^{(N)}(s)$ we write it in the form

$$J_3^{(N)}(s) = \int_{[0,N]} (1-e^{s\tau}\,\tilde{G}(s)) \int_{(\tau,\infty)} e^{-st}\,|\mu(t)|dt\; dG(\tau) =$$

$$\int_{[0,N]} (1-e^{s\tau}\,\tilde{G}(s))(\tilde{M}(s) - \int_{[0,\tau]} e^{-st}|\mu(t)|dt)dG(\tau) =$$

$$J_4^{(N)}(s) + J_5^{(N)}(s) \;,$$

where

$$J_4^{(N)}(s) = -\int_{[0,N]} (1-e^{s\tau}\,\tilde{G}(s)) \int_{[0,\tau]} e^{-st}|\mu(t)|dt\; dG(\tau) \;,$$

and

$$J_5^{(N)}(s) = \tilde{M}(s) \int_{[0,N]} (1-e^{s\tau}\,\tilde{G}(s))\; dG(\tau) \;.$$

Now,

$$|J_4^{(N)}(s)| \leq \int_{[0,N]} e^{-st}|\mu(t)|dt \int_{[0,N]} |(1-\tilde{G}(s))e^{s\tau} + 1-e^{s\tau}|dG(\tau) \leq$$

$$[(1-\tilde{G}(s))e^{sN} + e^{sN} - 1] \int_{[0,N]} |\mu(t)|dt \;.$$

Moreover, replacing $\tilde{G}(s)$ by $1-s\tilde{Q}(s)$, we find that

$$J_5^{(N)}(s) = s\tilde{M}(s) \int_{[0,N]} (\frac{1-e^{s\tau}}{s} + e^{s\tau}\,\tilde{Q}(s))dG(\tau) \;.$$

Since, for $0 \leq \tau \leq N$ and s sufficiently small, say $0 < s < s_0$, we have

$$|\,\frac{1-e^{s\tau}}{s} + e^{s\tau}\,\tilde{Q}(s)| \leq \frac{e^{sN}-1}{s} + e^{sN}\,\tilde{Q}(0) =$$

$$\frac{e^{sN}-1}{s} + e^{sN}\, m_G \leq C_N \;,$$

where C_N is independent of s, it follows from the Lebesgue bounded convergence theorem that

$$\limsup_{s \downarrow 0} |J_5^{(N)}(s)| \leq A \left| \int_{[0,N]} (-\tau + m_G) dG(\tau) \right| =$$

$$A \left| \int_{(N,\infty)} \tau \, dG(\tau) - m_G(1-G(N)) \right| .$$

Summarizing these estimates we have,

$$\tilde{M}(s) - \tilde{\Gamma}(s)\,\tilde{G}(s) = J_1(s) + J_2^{(N)}(s) + J_4^{(N)}(s) + J_5^{(N)}(s) \quad ,$$

$$|J_2^{(N)}(s)| \leq A \int_{[N,\infty)} t dG(t) \quad \text{for} \quad 0 < s \leq \frac{1}{N} \quad \text{and } N \geq 2m_G \; ,$$

$$|J_4^{(N)}(s)| \leq [(1-\tilde{G}(s))e^{sN} + e^{sN} - 1] \int_{[0,N]} |\mu(t)| dt \; ,$$

and

$$\limsup_{s \downarrow 0} |J_5^{(N)}(s)| \leq A \left| \int_{(N,\infty)} \tau \, dG(\tau) - m_G(1 - G(N)) \right| \quad .$$

It is clear from these results that

$$\lim_{s \downarrow 0} (\tilde{M}(s) - \tilde{\Gamma}'(s)\,\tilde{G}(s) - J_1(s)) = 0 \; .$$

But, by the monotone convergence theorem,

$$\lim_{s \downarrow 0} J_1(s) = \int_{[0,\infty)} |\mu(t)| \, Q(t) \, dt \; .$$

Hence,

$$\lim_{s \downarrow 0} (\tilde{M}(s) - \tilde{\Gamma}(s)\,\tilde{G}(s)) = \int_{[0,\infty)} |\mu(t)| Q(t) dt \geq 0 \; . \tag{17}$$

Comparing (10) and (17) we see that

$$(18) \qquad \int_{[0,\infty)} |\mu(t)|Q(t)\,dt = 0 ,$$

which, in turn, implies that

$$(19) \qquad \mu(t)\,Q(t) = 0 \quad \text{a.e. } [\lambda].$$

To prove that $\mu(x) = 0$ a.e. $[\lambda]$ we argue contrapositively. Assume that the set,

$$\{x : |\mu(x)| > 0\} ,$$

has positive Lebesgue measure. Let

$\sigma^* = \sup \{ t : \mu(x) = 0$ for all x in the interval $0 \leq x \leq t$, except possibly a set of Lebesgue measure $0 \}$. If the set in braces is empty, we define

$$\sigma^* = 0 .$$

Clearly, $0 \leq \sigma^* < \infty$, and for every $t > \sigma^*$, $|\mu(t)| > 0$ on a subset of $[\sigma^*, t]$ having positive Lebesgue measure. It follows from (19) and the properties of G that there exists a number q^* in the interval $0 \leq q^* \leq \sigma^*$ such that

$$Q(t) \begin{cases} = 0 & \text{for } t \geq q^* \\ > 0 & \text{otherwise.} \end{cases}$$

Put

$$\gamma(x) = \int_{[0,\infty)} \mu(x+t)dG(t) = \int_{[0,q^*]} \mu(x+t)dG(t) ,$$

$$\tilde{\gamma}(s) = \int_{[0,\infty)} e^{-st}\,\gamma(t)\,dt , \qquad s > 0 ,$$

and

$$\tilde{\mu}(s) = \int_{[0,\infty)} e^{-st}\,\mu(t)\,dt, \qquad s > 0 .$$

An easy calculation shows that

$$\tilde{\gamma}(s) = \tilde{\mu}(s) \int_{[0,q^*]} e^{st}\,dG(t) .$$

Moreover, from (3) and the convolution theorem for Laplace-Stieltjes transforms, it follows that

$$\tilde{\mu}(s) = \tilde{\gamma}(s)\,\tilde{F}(s) .$$

Hence,

$$(20)\quad \tilde{\mu}(s)(1-\tilde{F}(s)) \int_{[0,q^*]} e^{st}\,dG(t)) \equiv 0 \quad \text{for } s > 0 .$$

Since we have assumed $|\mu(x)| > 0$ on a set of positive Lebesgue measure, $\tilde{\mu}(s)$, being analytic, can have only a finite number of zeros in each bounded interval. It follows from (20) and the analyticity of the functions $\tilde{F}(s)$ and $\int_{[0,q^*]} e^{st}\,dG(t)$ that

$$(21)\quad \tilde{F}(s) \int_{[0,q^*]} e^{st}\,dG(t) \equiv 1 \quad \text{for } s > 0 .$$

Let ξ and η be independent random variables having, respectively, distribution functions $F(x)$ and $G(x)$. Then (21) can be expressed in the form

$$E[e^{-s\xi}]E[e^{s\eta}] = E[e^{-s(\xi-\eta)}] \equiv 1 \quad \text{for } s > 0 .$$

By the uniqueness theorem for Laplace-Stieltjes transforms, it follows that the distribution function of $\xi-\eta$ degenerates at the origin, i.e., $\xi = \eta$ almost surely. The independence of ξ and η

then implies that both are almost surely constant, i.e., F and G degenerate at the same point. This contradicts the hypothesis and completes the sufficiency part of the proof. The necessity part is proved in [2], where, under the assumption $\mu_F < \mu_G$, the existence is established of a non-decreasing solution $\phi(x)$ such that $\lim_{x \to \infty} \phi(x) = 1$.

It had been our conjecture, originally, that under the non-degeneracy assumption, if (5) holds, the only bounded solution of (3) is the trivial one. The preceding theorem falls a little short of establishing this, and whether or not it is the case is still an open question. It is clear, however, from (3) and theorem 1 that, if either F or G is absolutely continuous, and (5) holds, the conjecture is correct. Indeed, if either F or G is absolutely continuous, any solution of (3) that is zero a.e. [λ] is identically zero, regardless of the relation between m_F and m_G. This might lead one to suspect that it is possible to deduce the conjectured result from the conclusion of theorem 1 without further consideration of the relation between m_F and m_G. That this is not the case is demonstrated by the following example:

Let F and G have unit jumps at, respectively, x_0 and $x_0 + 1$. The function,

$$\mu(x) = \begin{cases} 0 & \text{for } x < x_0 \\ 1 & \text{for } x = x_0 \\ \mu(x-1) & \text{for } x > x_0 \end{cases},$$

is zero a.e. [λ], bounded, and satisfies (3), but it is not identically zero. Of course in this example $\mu_F < \mu_G$, so that it

does not contradict the conjecture. But it does establish that it is hopeless to try to pass from the conclusion of theorem 1 to the stronger conclusion that $\mu(x) \equiv 0$ without invoking the relation between the means.

Partial results in the direction of the conjecture are given by the observation that it is true, as we have seen, if at least one of the governing distributions is absolutely continuous, and it is true, also, under the hypotheses of theorem 2, which is stated below. For the proof of this result, we need the following lemma, which may be of independent interest.

LEMMA. If $\lim_{t \to \infty} F(t)G(t) < 1$, then $\mathcal{C}^*$ contains only the trivial solution.

Proof : Put

$$L = \lim_{t \to \infty} F(t)G(t) .$$

Inserting the bound $|\mu(x)| \leq A$ into (3), we see that

$$|\mu(x)| \leq AL .$$

Similarly, after n iterations, we find

$$|\mu(x)| \leq AL^n ,$$

which, by a passage to the limit, establishes the lemma.

THEOREM 2. Assume that $m_G < \infty$, and F and G do not degenerate at a common point. If neither F nor G has a singular component, and the union of their discontinuity sets is a subset of an arithmetic progression, then (5) is a necessary and sufficient condition that the only bounded solution of (3) is $\mu(x) \equiv 0$.

Proof : By hypothesis and the standard decomposition theorem for distribution functions, F and G can be written in the form

$$F = F_d + F_{ac}$$

and

$$G = G_d + G_{ac} ,$$

where the subscripts d and ac indicate, respectively, the discrete and absolutely continuous components of the distribution functions involved. Assume that (5) holds. Let $\mu(x)$ be any bounded solution of (3). Since $\mu(x) = 0$ a.e. $[\lambda]$, (3) is equivalent to

$$(22) \qquad \mu(x) = \int_{[0,\infty)} \int_{[0,x]} \mu(x-t+\tau)\, dF_d dG_d \;.$$

Let

$$a_n = a + nd , \quad d > 0, \quad n = 0,1, \ldots$$

be the arithmetic progression of points at which the jumps of F and G occur, and put

$$f_n = F(a_n) - F(a_n - 0)$$

and

$$g_n = G(a_n) - G(a_n - 0) .$$

Let θ be an arbitrary number in the interval $0 \leq \theta < d$, and put

$$\nu(n) = \mu(a_n + \theta) , \quad n = 0,1, \ldots .$$

Since θ is arbitrary, to prove the sufficiency statement of the theorem it suffices to show that $\nu(n) = 0$ for $n = 0,1,\ldots$.

Now, from (22), we have

$$(23)\quad \nu(n) = \mu(a_n + \theta) = \sum_{j=0}^{\infty} g_j \sum_{k=0}^{n} f_k \mu(a_n - a_k + a_j + \theta) =$$

$$\sum_{j=0}^{\infty} g_j \sum_{k=0}^{n} f_k \, \nu(n+j-k) \ .$$

Comparing (23) with (3) we see that ν satisfies (3) at points $x = 0, 1, \ldots$, the governing distribution functions F^* and G^* being step functions with jumps

$$f_n = F^*(n) - F^*(n-0)$$

and

$$g_n = G^*(n) - G^*(n-0), \quad n = 0, 1, \ldots \ .$$

If either $\sum_{n=0}^{\infty} f_n < 1$ or $\sum_{n=0}^{\infty} g_n < 1$, then from the lemma it follows that $\nu(n) \equiv 0$. If both $\sum_{n=0}^{\infty} f_n = 1$ and $\sum_{n=0}^{\infty} g_n = 1$, then $F^*_{ac} \equiv G^*_{ac} \equiv 0$. In this case,

$$\mu_F = \sum_{n=0}^{\infty} a_n f_n = a + d \sum_{n=0}^{\infty} n \, f_n \geq \mu_G = \sum_{n=0}^{\infty} a_n g_n = a + d \sum_{n=0}^{\infty} n g_n \ ,$$

from which it follows that $\mu_{G^*} \leq \mu_{F^*}$. The argument in theorem 1 which led to (18) then shows that

$$(24)\quad \sum_{n=0}^{\infty} |\nu(n)| Q^*(n) = 0 \ ,$$

where

$$Q^*(n) = 1 - G^*(n) = \sum_{j=n+1}^{\infty} g_j \ .$$

We now argue contrapositively. Suppose $\nu(n) \not\equiv 0$. Let

$$N = \min \{ n : |\nu(n)| > 0 \} \ .$$

Then,

$$\nu(n) = 0, \quad 0 \leq n < N \ ,$$

and

$$|\nu(N)| > 0 .$$

From (23) it follows that

$$(25) \quad \sum_{n=0}^{\infty} \nu(n)x^n = \sum_{n=N}^{\infty} \nu(n)x^n = \sum_{m=0}^{\infty} f_m x^m \sum_{j=0}^{\infty} g_j \sum_{n=0}^{\infty} \nu(n+j)x^n \quad .$$

Putting

$$(26) \qquad \omega(n) = \sum_{j=0}^{\infty} g_j \nu(n+j)$$

and interchanging the order of summation in (25) (which is justified by the absolute convergence of the series involved), we obtain

$$(27) \quad \sum_{n=N}^{\infty} \nu(n)x^n = \sum_{m=0}^{\infty} f_m x^m \sum_{n=0}^{\infty} \omega(n)\, x^n \quad .$$

Since $|\nu(N)| > 0$, it follows from (24) that $Q^*(N) = 0$. Hence, by by the monotonicity of Q^*, there exists a number N' in the interval $0 \leq N' \leq N$ such that

$$Q^*(n) > 0 \quad , \quad 0 \leq n < N' \quad ,$$

and

$$Q^*(n) = 0 \quad , \quad n \geq N' \quad ,$$

i.e.,

$$g_{N'} > 0 \quad ,$$

and

$$g_j = 0 \quad , \quad j \geq N' + 1 \; .$$

Thus, (26) reduces to

$$\omega(n) = \sum_{j=0}^{N'} g_j \nu(n+j) \; .$$

Moreover, since $\nu(n) = 0$ for $0 \leq n < N$,

$$\omega(n) = 0 , \quad 0 \leq n \leq N-N'-1 ,$$

and

$$\omega(N-N') = g_{N'} \nu(N) .$$

Equation (27) then takes the form

$$\sum_{n=N}^{\infty} \nu(n)x^n = \sum_{m=0}^{\infty} f_m x^m \sum_{n=N-N'}^{\infty} \omega(n)x^n ,$$

from which it follows that

$$\nu(N) = f_{N'} \omega(N-N') = f_{N'} g_{N'} \nu(N) .$$

Since $\nu(N) \neq 0$, this implies

$$f_{N'} g_{N'} = 1 ,$$

which contradicts the non-degeneracy assumption and completes the sufficiency part of the proof. The necessity part, as was pointed out in the proof of theorem 1, is contained in [2].

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